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Injective envelopes and flat covers of Matlis reflexive modules

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Abstract

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We show that for a commutative noetherian local ring R, every Matlis reflexive R-module has a reflexive injective envelope if and only if every Matlis reflexive R-module has a reflexive flat cover. This occurs if and only if R is complete and has Krull dimension less than or equal to 1. We also exhibit a family of Matlis reflexive R-modules whose injective envelopes are not Matlis reflexive.

1. Introduction

In this paper R will denote a commutative noetherian ring with identity. If M is an R-module, then $E_R(M)$ or just $E(M)$ will denote an injective envelope of M as an R-module.

The notion of a flat cover of a module was introduced by Enochs in [5]. A flat cover of an R-module M consists of a flat module F and a linear map $\sigma : F \rightarrow M$ with the following two properties:

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(i) For any flat module F' and any linear map $\sigma' : F' \to M$ there is a linear map $g : F' \to F$ such that

$$\begin{tikzcd} & F' \arrow[dl, dotted, "g"'] \arrow[d, "\sigma'"] \\ F \arrow[r, "\sigma"] & M \end{tikzcd}$$

commutes.

(ii) Any linear $g : F \to F$ such that

$$\begin{tikzcd} & F \arrow[dl, dotted, "g"'] \arrow[d, "\sigma"] \\ F \arrow[r, "\sigma"] & M \end{tikzcd}$$

commutes is an automorphism.

If $\sigma : F \to M$ satisfies (i) it is called a flat precover. Clearly such a σ is always reflexive. In [6, Proposition 1.2, p. 180] it was shown that if R is a complete local ring, then every Matlis reflexive R-module has a flat cover.

For a local ring $(R, \mathfrak{m})$ with residue field $k = R/\mathfrak{m}$, let $E = E_R(k)$. Then M^v denotes the Matlis dual $\mathrm{Hom}_R(M, E)$ of the R-module M, and M is said to be Matlis reflexive if the canonical map $\tau_M : M \to M^{vv}$ is an isomorphism. It is easy to check directly that τ_M is always injective. In this paper 'reflexive' will always mean Matlis reflexive. We need the following results of Matlis.

(a) Let $\hat{R}$ be the $\mathfrak{m}$-adic completion of R. Then by [9, Theorem 3.7, p. 522],

$$\mathrm{Hom}_R(E, E) \cong \hat{R} .$$

(b) Let R be a complete local ring. Then finitely generated and artinian R-modules are reflexive. Also, M is an artinian (noetherian) R-module if and only if M^v is noetherian (artinian) [9, Corollary 4.3, p. 528]. In particular, $R^v \cong E$ is artinian, $E^v \cong R$ and both are reflexive.

The main results of Matlis [9] are nicely summarized in Matsumara's book [11]. See also the book by Sharpe and Vámos [13].

We note for use later the following general 'change of ring' facts. Let R be any ring and let $I \subset R$ be an ideal. Given an R/I-module M, we can regard M canonically as an R-module. Suppose now that E is an injective R-module. Let

$$\mathrm{ann}_E(I) = \{x \in E \mid Ix = 0\} .$$

Then $\mathrm{ann}_E(I)$ is a submodule of E, and it is clear that $\mathrm{ann}_E(I)$ is also an R/I-module. It is not hard to see that $\mathrm{ann}_E(I)$ is an injective R/I-module.

Lemma 1.1. *Suppose M is an R/I-module and $E = E_R(M)$. Then*

$$E_{R/I}(M) = \operatorname{ann}_E(I) .$$

Proof. Since M is an R/I-module, $M \subset \operatorname{ann}_E(I)$. We have just noted that $\operatorname{ann}_E(I)$ is an injective R/I-module. It remains to be shown that $\operatorname{ann}_E(I)$ is an essential extension of M as an R/I-module. But this is true since as R-modules it is clear that $\operatorname{ann}_E(I)$ is an essential extension of M. □

In particular, for a local ring $(R, \mathfrak{m})$ with residue field $k = R/\mathfrak{m}$, if $E = E_R(k)$, then $E_{R/I}(k) = \operatorname{ann}_E(I)$. But if M is an R/I-module and is regarded canonically as an R-module, then

$$\operatorname{Hom}_R(M, E) = \operatorname{Hom}_{R/I}(M, E_{R/I}(k)) ,$$

and so $M^\vee$ is unambiguous. This means that we can think of M as an R-module or as an R/I-module and $M^\vee$ means the same thing. Also note that $IM^\vee = 0$, so $M^{\vee\vee}$ is unambiguous. This means then that such a module M is reflexive as an R-module if and only if it is reflexive as an R/I-module.

2. The injective envelope of a Matlis reflexive module

Let $(R, \mathfrak{m})$ be a local ring. If M is an R-module and $S \subset M$ a submodule, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d, "\tau_S"] & M \arrow[r] \arrow[d, "\tau_M"] & (M/S) \arrow[r] \arrow[d, "\tau_{(M/S)}"] & 0 \\
0 \arrow[r] & S^{\vee\vee} \arrow[r] & M^{\vee\vee} \arrow[r] & (M/S)^{\vee\vee} \arrow[r] & 0
\end{tikzcd}$$

with injective vertical maps. A diagram chase using the snake lemma shows that M is reflexive if and only if S and M/S are reflexive. A finite direct sum $\bigoplus M_i$ is reflexive if and only if each M_i is reflexive. In particular, if $E(M)$ is reflexive, then M, being a submodule, is also reflexive. But we note that the converse is not true in general. For example, if k is a field then $R = k[[x, y]]$ is a complete local ring and so is Matlis reflexive. We will show that $E(R)$, the field of fractions of R, is not a Matlis reflexive R-module.

Given a reflexive module M, in this section we establish conditions under which the injective envelope $E(M)$ is also reflexive module.

Proposition 2.1. *If $(R, \mathfrak{m})$ is a complete local ring and M is an artinian R-module, then $E(M)$ is reflexive.*

Proof. We have already noted that if $k = R/\mathfrak{m}$, then $E(k)$ is reflexive. Since $E(M) \cong E(k)^n$ for some n [9, Proof of Corollary 4.3], it follows that $E(M)$ is reflexive. □

Lemma 2.2. *Let R be any local ring, M an R-module, and $S \subset M$ a submodule. If $E(S)$ and $E(M/S)$ are both reflexive, then $E(M)$ is reflexive.*

Proof. There is an obvious injection $M \to E(S) \oplus E(M/S)$ which makes the diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dotted] & M/S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E(S) \arrow[r] & E(S) \oplus E(M/S) \arrow[r] & E(M/S) \arrow[r] & 0
\end{tikzcd}$$

commute. It follows that there is an injection $E(M) \to E(S) \oplus E(M/S)$. Thus $E(M)$ is a direct summand of a reflexive module and therefore $E(M)$ is reflexive. □

We begin by assuming that R is a domain.

Proposition 2.3. *Suppose R is a complete local domain and* $\dim R = 1$. *Then M is a reflexive R-module if and only if $E(M)$ is reflexive.*

Proof. We only need to show that if M is reflexive, then $E(M)$ is reflexive. For an R-module M, let tM denote the torsion submodule of M. Replacing S in Lemma 2.2 by tM, we get that $E(M)$ is a direct summand of $E(tM) \oplus E(M/tM)$. Since M is reflexive, the submodule tM is as well, hence so is M/tM. So it is enough to show that $E(M)$ is reflexive in both cases—M torsion and M torsion-free. We do this using the following lemmas.

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a complete local domain with* $\dim R = 1$. *If S is a finitely generated torsion-free R-module, then $E(S)$ is reflexive.*

Proof. We will denote the field of fractions of R by Q and will let $k = R/\mathfrak{m}$ be the residue field. Then S embeds in a finitely generated free R-module, say R^n for some n [12, Lemma 4.31]. Noting that $Q = E(R)$, we have $E(S) \subset Q^n$, so it is enough to show that Q is reflexive. And since R is a finitely generated R-module, to show that Q is reflexive it is enough to show that Q/R is artinian [9, Corollary 4.3 and Snake Lemma]. But this follows from [10, Theorem 1]. □

Lemma 2.5. *Let R be a complete local domain with* $\dim R = 1$. *If M is a torsion-free R-module and is reflexive, then $E(M)$ is reflexive.*

Proof. Since M is reflexive, by [6, Proposition 1.3, p. 181] there exists a finitely generated submodule $S \subset M$ with M/S artinian. But $E(M/S)$ is reflexive by Proposition 2.1, and $E(S)$ is reflexive by Lemma 2.4. Therefore, $E(M)$ is reflexive by Lemma 2.2. $\square$

Lemma 2.6. *Let R be a complete local domain with* $\dim R = 1$. *If S is a finitely generated torsion R-module, then S is artinian and so $E(S)$ is reflexive.*

Proof. We first consider the case where $S = Rx$ is cyclic. Then $\operatorname{ann}(x) \neq 0$ since S is torsion, hence $\dim(R/\operatorname{ann}(x)) = 0$, and so $R/\operatorname{ann}(x) \cong S$ is artinian. Therefore, $E(S)$ is reflexive by Proposition 2.1.

Next suppose that $S = Rx_1 + \cdots + Rx_t$, and let $T = Rx_1 \oplus \cdots \oplus Rx_t$. There is an obvious surjection $\phi : T \to S$, but T is artinian by the above argument, and therefore S is artinian. $\square$

Lemma 2.7. *Let R be a complete local domain with* $\dim R = 1$. *If M is a torsion R-module and is reflexive, then $E(M)$ is reflexive.*

Proof. Since M is reflexive, we have a finitely generated submodule $S \subset M$ with M/S artinian. Lemma 2.6 and Proposition 2.1 show that $E(S)$ and $E(M/S)$ are reflexive, and so $E(M)$ is reflexive by Lemma 2.2. $\square$

This completes the proof of Proposition 2.3. $\square$

Proposition 2.8. *Suppose R is a complete local domain with field of fractions Q. Then the following statements are equivalent*:

(1) *Q is reflexive.*
(2) *Q/R is artinian.*
(3) $\dim R \leq 1$.

Proof. (1)$\Leftrightarrow$(2) If Q/R is artinian, then Q has a finitely generated submodule—namely R—with an artinian quotient, and so Q is reflexive.

Conversely, if Q/R is not artinian then let $S \subset Q$ be any finitely generated submodule. To show that Q is not reflexive it suffices by [6, Proposition 1.3] to show that Q/S is not artinian.

Suppose that S is generated by $s_1, \ldots, s_n$. Then for each i we have $s_i = f_i/g_i$, where $f_i, g_i \in R$, and $g_i \neq 0$. Setting $y = 1/(g_1 \cdots g_n)$, we have that $S \subset Ry$. But the map $R \to Ry$ sending 1 to $1y$ is bijective, and so $R \cong Ry$. Hence there is a well-defined epimorphism $\phi : Q/S \to Q/R$. So Q/S is not artinian.

(2)$\Leftrightarrow$(3) This follows from [10, Theorem 1, p. 571]. $\square$

Remark. The proposition supplies plenty of examples of reflexive modules whose injective envelopes are not reflexive—any complete local domain of dimension at least two (regarded as a module over itself) fits the bill.

Now we drop the assumption that R is a domain.

Theorem 2.9. *Suppose* $(R, \mathfrak{m})$ *is a complete local ring with* $\dim R \leq 1$. *Then an* R*-module* M *is reflexive if and only if* $E(M)$ *is reflexive.*

Proof. First assume $\dim R = 0$. If M is reflexive, then there is a finitely generated submodule $S \subset M$ such that M/S is artinian. But then the finitely generated module S is also an artinian module since R is an artinian ring. So both $E(S)$ and $E(M/S)$ are reflexive by Proposition 2.1, and therefore $E(M)$ is reflexive.

Now assume $\dim R = 1$. Let M be a reflexive R-module. Then for any finitely generated R-module S we can write

$$E(S) = \bigoplus_{\mathfrak{p} \in \operatorname{Spec}(R)} \mu_0(\mathfrak{p}, S) E(R/\mathfrak{p}) , \tag{1}$$

where $\mu_0(\mathfrak{p}, S) < \infty$ for all primes $\mathfrak{p}$ [3, Lemma 2.7, p. 11]. Since the prime ideals of R are the maximal ideal $\mathfrak{m}$ and a finite number of minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$, (1) is a finite direct sum. So to prove that $E(S)$ is reflexive it is enough to prove that $E(R/\mathfrak{p})$ is reflexive whenever $\mathfrak{p}$ is a minimal prime ideal of R. But if $\mathfrak{p}$ is minimal, then $\dim R_{\mathfrak{p}} = 0$, so in this case $R_{\mathfrak{p}}$ is an artinian local ring with residue field $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$. From [9] (or [11, Theorem 18.4(vi)]) we know that

$$E_R(R/\mathfrak{p}) = E_{R_{\mathfrak{p}}}(k(\mathfrak{p})) .$$

Since the Matlis dual of an artinian is finitely generated [9, Corollary 4.3], we have that $R_{\mathfrak{p}}^{v} \cong E_{R_{\mathfrak{p}}}(k(\mathfrak{p})) \cong E_R(R/\mathfrak{p})$ is a finitely generated $R_{\mathfrak{p}}$-module. And since $\dim R_{\mathfrak{p}} = 0$, $E_R(R/\mathfrak{p})$ has finite length over $R_{\mathfrak{p}}$. This means that there is a composition series for $E_R(R/\mathfrak{p})$,

$$0 \subset A_0 \subset \cdots \subset A_n = E_R(R/\mathfrak{p}) , \tag{2}$$

where each factor A_i/A_{i-1} is a simple $R_{\mathfrak{p}}$-module. So each factor must be $k(\mathfrak{p})$. Recall that $k(\mathfrak{p})$ is naturally isomorphic to Q, the field of fractions of the domain $R/\mathfrak{p}$. Since $Q = E_{R/\mathfrak{p}}(R/\mathfrak{p})$ is a reflexive $R/\mathfrak{p}$-module by the domain case (Proposition 2.8), Q is also reflexive as an R-module. Because each factor in the composition series (2) is reflexive as an R-module, $E_R(R/\mathfrak{p})$ is reflexive as an R-module.

This shows that $E(S)$ is reflexive if $S \subset M$ is any finitely generated submodule of M with M/S artinian. But $E(M/S)$ is also reflexive by Proposition 2.1. Hence by Lemma 2.2, $E(M)$ is reflexive. □

3. The flat cover of a Matlis reflexive module

Let $(R, \mathfrak{m})$ be a complete local ring. We have already noted that every Matlis reflexive R-module has a flat cover [6, Proposition 1.2]. We now ask whether the flat cover of a reflexive module is reflexive. Because R is a noetherian local ring, it is semi-perfect and every finitely generated R-module has a projective cover ([2, Theorem 2.1] or [12, pp. 135–136]).

Proposition 3.1. *Let $\phi : P \to M \to 0$ be the projective cover of the finitely generated R-module M. Then (ϕ, P) is the flat cover of M.*

Proof. (i) For any flat R-module F and homomorphism $f : F \to M$, we have the diagram

$$\begin{tikzcd}
& F^{\vee} & \\
P^{\vee} \arrow[ur, dotted, "g"] & M^{\vee} \arrow[l, "\phi^*"] \arrow[u, "f^*"'] & 0, \arrow[l]
\end{tikzcd}$$

where ϕ^* is an injection, and (by [7] or [8, Exercise 8(b), p. 614]) $F^{\vee}$ is an injective module. Hence there is a homomorphism $g : P^{\vee} \to F^{\vee}$ such that $g\phi^* = f^*$. Note that P is finitely generated and P, M are reflexive. We have the diagram

$$\begin{tikzcd}
& & F^{\vee\vee} \arrow[ddll, "g^*"'] \arrow[dd, "f^{**}"] & \\
& & & F \arrow[ul, "\tau_F"'] \arrow[dl, dash] \arrow[dd, "f"] \\
P^{\vee\vee} \arrow[rr, "\phi^{**}"] & & M^{\vee\vee} \arrow[dl, "g_1"] & \\
& P \arrow[ul, "\tau_P"] \arrow[rr, "\phi"'] & & M \arrow[ul, "\tau_M"']
\end{tikzcd}$$

We know that τ_P and τ_M are isomorphisms. Set $g_1 = \tau_P^{-1} g^* \tau_F$. A diagram chase shows that $f = \phi g_1$. Therefore, $\phi : P \to M \to 0$ is a flat precover of M.

(ii) For any endomorphism $f : P \to P$ such that $\phi = \phi f$, we need to prove that f is an automorphism of P. But $\ker(\phi)$ is a superfluous submodule of P, and so f is epic. Hence f is an automorphism because P is a noetherian module. $\square$

We will need the following result due to Ishikawa.

Lemma 3.2. *If E and E' are injective R-modules, then* $\operatorname{Hom}_R(E, E')$ *is a flat R-module.*

Proof. See [7, Theorem 1.5, p. 293]. □

For the general case, we have the following result.

Proposition 3.3. *Let M be a reflexive, $F \xrightarrow{\phi} M$ be the flat cover of M, and let $\tau_F : F \to F^{\vee\vee}$ be the canonical injection. Then F is a direct summand of $F^{\vee\vee}$.*

Proof. Since F is the flat cover of M, ϕ is epic. We have the following diagrams:

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, "\tau_F"'] & M \arrow[d, "\tau_M"] \\ F^{\vee\vee} \arrow[r, "\phi^{**}"] & M^{\vee\vee} \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\tau_F"] & F^{\vee\vee} \arrow[d, "\tau_M^{-1}\phi^{**}"] \arrow[dl, dotted, "\beta"'] & \\ F \arrow[r, "\phi"'] & M \arrow[r] & 0 \end{tikzcd}$$

Since $F^{\vee\vee}$ is flat (Lemma 3.2) and $\phi : F \to M$ is the flat cover, there exists a homomorphism $\beta : F^{\vee\vee} \to F$ such that $\tau_M^{-1}\phi^{**} = \phi\beta$, and then $\phi\beta\tau_F = \tau_M^{-1}\phi^{**}\tau_F = \phi$. Therefore, $\beta\tau_F : F \to F$ is an automorphism of F and F is isomorphic to a direct summand of $F^{\vee\vee}$. □

4. The main result

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a noetherian local ring. Then the following statements are equivalent:*

(1) *Every reflexive R-module has a reflexive injective envelope.*

(2) *Every reflexive R-module has a reflexive flat cover.*

(3) *R is complete and* $\dim R \leq 1$.

Furthermore, suppose these conditions hold and M is a reflexive R-module. Let E be injective and F flat. Then $M \to E$ is an injective envelope of M if and only if $E^\vee \to M^\vee$ is a flat cover of $M^\vee$. Similarly, $F \to M$ is a flat cover of M if and only if $M^\vee \to F^\vee$ is an injective envelope of $M^\vee$.

Proof. (1)$\Rightarrow$(2) Let M be reflexive. Then $M^\vee$ and $E := E(M^\vee)$ are reflexive. Let $0 \to M^\vee \xrightarrow{h} E$ be the injective envelope of $M^\vee$. Let F be any flat R-module and let $f : F \to M^{\vee\vee}$ be an R-module homomorphism. Then we have the following diagram

$$\begin{tikzcd} & F \arrow[d, "f"] \arrow[dl, dotted, "g_1"'] & \\ E^\vee \arrow[r, "\phi"] & M^{\vee\vee} \arrow[r] & 0 \end{tikzcd}$$

where $\phi = h^*$, and there exists a linear map g_1 such that $\phi g_1 = f$. Since E^ν is flat by Lemma 3.2, this shows that (E^ν, ϕ) is a flat precover of $M^{\nu\nu}$.

For any linear map $g : E^\nu \to E^\nu$ such that $\phi = \phi g$, we will prove that g is an automorphism of E^ν. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M^\nu \arrow[r, "h"] \arrow[d, "\tau_{M^\nu}"', "\cong"] & E \arrow[d, "\tau_E"', "\cong"] \\
& M^{\nu\nu\nu} \arrow[r, "h^{**}"] \arrow[d, "\phi^* = h^{**}"'] & E^{\nu\nu} \arrow[dl, "g^*"] \\
& E^{\nu\nu} &
\end{tikzcd}$$

We have that $h^{**}\tau_{M^\nu} = g^*\tau_E h$. Therefore, $\tau_E h = g^*\tau_E h$, and $h = \tau_E^{-1} g^*\tau_E h$. Then $\tau_E^{-1} g^* \tau_E$ is an automorphism of E and g^* is an automorphism of $E^{\nu\nu}$. But then g is an automorphism of E^ν. This proves $(1) \Rightarrow (2)$.

In showing $(2) \Rightarrow (3)$ we first show $(2) \Rightarrow R$ is complete. Now let E denote $E(k)$, where $k = R/\mathfrak{m}$. Note that $k \cong \mathrm{Hom}_R(k, E) = k^\nu$ and hence k has a flat cover $F \xrightarrow{\phi} k \to 0$ by [6, Proposition 1.1]. Since F is flat, F^ν is injective; and both F and F^ν are reflexive. We have the diagram

$$\begin{tikzcd}
0 \arrow[r] & k^\nu \cong k \arrow[r] \arrow[d, "\phi^*"'] & E \arrow[dl, "\alpha"] \\
& F^\nu &
\end{tikzcd}$$

and α is monic. We can think of E as a submodule of F^ν. Now consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r, "\alpha"] \arrow[d, "\tau_E"] & F^\nu \arrow[r, "\beta"] \arrow[d, "\tau_{F^\nu}"] & X \arrow[r] \arrow[d, "\tau_X"] & 0 \\
0 \arrow[r] & E^{\nu\nu} \arrow[r, "\bar{\alpha}"] & F^{\nu\nu\nu} \arrow[r, "\bar{\beta}"] & X^{\nu\nu} \arrow[r] & 0
\end{tikzcd}$$

Since τ_{F^ν} and $\bar{\beta}$ are surjective, τ_X is surjective and so τ_X is an isomorphism. It follows that τ_E is an isomorphism. Hence E is reflexive. But $R^\nu \cong E$, and since R^ν is reflexive, R is reflexive [4, Proposition 6]. It follows form [9, Theorem 3.7] that R is complete.

Next we show that if R is a complete local ring and every reflexive module has a reflexive flat cover, then $\dim R \leq 1$.

Case 1. Assume that R is a domain. Because E is reflexive, so is F, where

$F \xrightarrow{\phi} E$ is the flat cover of E. We know that ϕ is epic. We have an injection $R \xrightarrow{\cong} E^{\nu} \xrightarrow{\phi^*} F^{\nu}$, and F^{ν} is injective and reflexive. It follows that $E(R)$ is a submodule of F^{ν}, and so $E(R)$ is reflexive. Let $Q = E(R)$, the field of fraction of R. By Proposition 2.8, $K := Q/R$ is artinian.

If $\mathfrak{m} = 0$ or $\mathfrak{m}$ is nonzero and principal, we are done. So we may assume $a_1, a_2, \ldots, a_n, b$ generate $\mathfrak{m}$, with $n \geq 1$ and $b \neq 0$. Consider, for each i, the descending chain

$$\left(\frac{a_i}{b} + R\right) \supset \left(\frac{a_i^2}{b} + R\right) \supset \cdots \supset \left(\frac{a_i^s}{b} + R\right) \supset \cdots$$

of submodules of K. Since K is an artinian R-module, there exists an integer s such that

$$\left(\frac{a_i^s}{b} + R\right) = \left(\frac{a_i^{s+1}}{b} + R\right).$$

That is,

$$\frac{a_i^s}{b} + R = r\left(\frac{a_i^{s+1}}{b} + R\right)$$

for some $r \in R$, and so

$$\frac{a_i^s}{b} = \frac{ra_i^{s+1}}{b} + c$$

for some $c \in R$. Hence $a_i^s(1 - ra_i) = bc$, and $a_i^s \in (b)$ since $1 - ra_i$ is invertible. From this we can get a sufficiently large integer t so that

$$\mathfrak{m}^t = (Ra_1 + \cdots + Rb)^t \subset (b).$$

So $\mathfrak{m}$ is the radical of a nonzero principal ideal, hence $\dim R = 1$. In [1] this is Theorem 11.14.

Case 2. If R is not a domain, let $\mathfrak{p}$ be a minimal prime ideal of R, and denote the domain (and module) $R/\mathfrak{p}$ by R_1. Now we are going to prove that the quotient field $Q_1 = E_{R_1}(R_1)$ is a reflexive R_1-module.

Since R is a reflexive R-module, so is R_1. If we replace R by R_1 and repeat the argument given in Case 1 mutatis mutandis, then we see that $E_1 := E_R(R_1)$ is a reflexive R-module. But then we know by Lemma 1.1 that

$$Q_1 = \operatorname{ann}_{E_1}(\mathfrak{p}) \subset E_1.$$

It follows that Q_1 is a reflexive R-module, and therefore Q_1 is a reflexive R_1-module. By Case 1 we know that $\dim R_1 \leq 1$, and so $\dim R \leq 1$.

$(3) \Rightarrow (1)$ By Theorem 2.9.

The last part of the theorem follows by chasing the obvious diagrams, Proposition 3.3, and the fact that for any injective module E, E is a direct summand of $E^{\vee\vee}$. $\square$

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